Q1 [14pts]

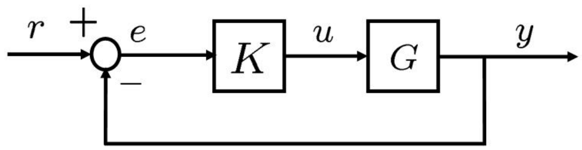


Figure 1: A feedback interconnection

Consider the feedback interconnection shown in Figure 1 where G and K are proper transfer functions. Here it is assumed that a transfer function is of the form $\frac{n(s)}{d(s)}$ where n and d are polynomials in s with no common factors. For the following questions answer if the statement is true or false. If true provide an example and if false provide a proof.

1. [7pts] There exist single-input, single-output proper transfer functions G and K such that $\frac{1}{1+GK}$ and $\frac{G}{1+GK}$ are stable but $\frac{K}{1+GK}$ is not.

   Solution: Let $G = \frac{s-1}{(s+1)^2}$ and $K = \frac{s+1}{s-1}$. Here $\frac{1}{1+GK} = \frac{s+1}{s+2}$, $\frac{G}{1+GK} = \frac{s-1}{(s+1)(s+2)}$ are stable but $\frac{K}{1+GK} = \frac{(s+1)^2}{(s-1)(s+2)}$ is not.

2. [7pts] There exist single-input, single-output proper transfer functions G and K such that $\frac{K}{1+GK}$ and $\frac{G}{1+GK}$ are stable but $\frac{1}{1+GK}$ is not.

   Solution: Suppose $G = \frac{n_g(s)}{d_g(s)}$ and $K = \frac{n_k(s)}{d_k(s)}$ where n_g, d_g, n_k and d_k are polynomials in s. Then it follows that

   - $\frac{1}{1+GK} = \frac{d_g d_k}{n_g n_k + d_g d_k}$,
   - $\frac{G}{1+GK} = \frac{n_g d_k}{n_g n_k + d_g d_k}$,
   - $\frac{K}{1+GK} = \frac{n_k d_g}{n_g n_k + d_g d_k}$,

   $\frac{1}{1+GK} = \frac{d_g d_k}{n_g n_k + d_g d_k}$, unstable implies that there is a s_0 in the right half plane (rhp) that $(n_g n_k + d_g d_k)(s_0) = 0$. Given that $\frac{G}{1+GK} = \frac{n_g d_k}{n_g n_k + d_g d_k}$, and $\frac{K}{1+GK} = \frac{n_k d_g}{n_g n_k + d_g d_k}$, are stable it follows that

   $$(n_g d_k)(s_0) = (n_k d_g)(s_0) = 0.$$

   As $(n_g d_k)(s_0) = 0$ there are two cases

   Case 1: Suppose $n_g(s_0) = 0$

   Then $d_g(s_0) \neq 0$ and thus from $(n_k d_g)(s_0) = 0$, $n_k(s_0) = 0$. Thus $d_k(s_0) \neq 0$. Thus

   $$n_g(s_0)n_k(s_0) + d_g(s_0)d_k(s_0) = d_g(s_0)d_k(s_0) \neq 0$$

   which is a contradiction.

   Case 2: Suppose $d_k(s_0) = 0$.

   Then $n_k(s_0) \neq 0$. Thus from $(n_k d_g)(s_0) = 0$, $d_g(s_0) = 0$. Thus $n_g(s_0) \neq 0$ and thus

   $$n_g(s_0)n_k(s_0) + d_g(s_0)d_k(s_0) = n_g(s_0)n_k(s_0) \neq 0$$

   which is a contradiction.

Q2 [15pts]

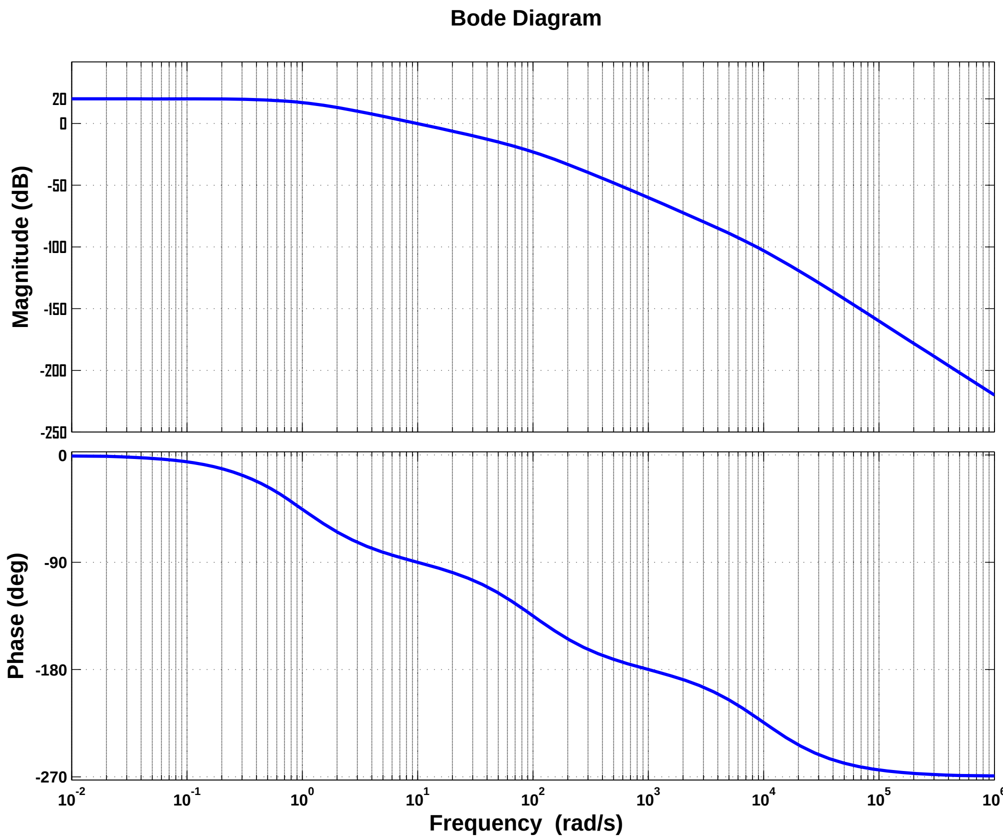


Figure 2: Bode-plot for Problem 2

Consider the bode plot of a minimum-phase transfer function $G(s)$ (the bode plot shows in the magnitude plot $20\log_{10}|G(j\omega)|$ in db on the Y axis).

1. [3pts] Draw the asymptotes on the bode plot. Use the asymptotes to determine the transfer function $G(s)$.

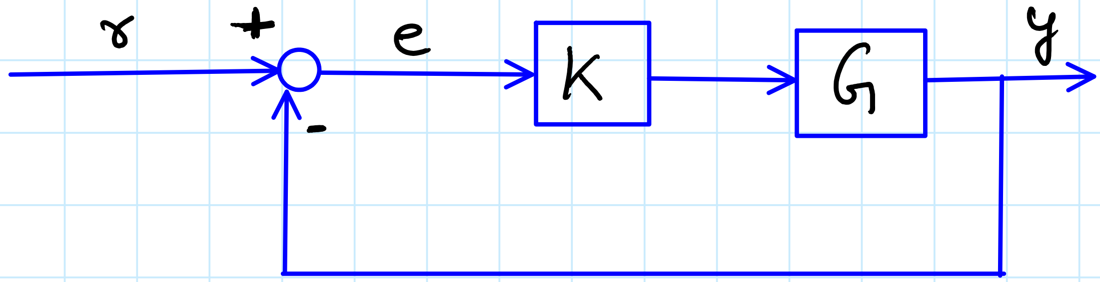
2. [3pts] (a)Determine the gain-crossover frequency (ω_{gc}) and the phase-crossover frequency (ω_{180}). (b) Determine the phase and gain margin.

3. [3pts] Suppose the plant G is in a unity negative feedback interconnection with a controller K (see Figure 1). With the controller $K = k_p$ a positive real constant, find the smallest value of k_p such that the interconnection shown is unstable. (Hint: Use the gain margin to obtain the result).

4. [3pts] With $K = 1$ determine the steady state error due to a step input for the interconnection shown. Also, determine the steady state error due to a ramp input.

5. [3pts] Design a Proportional Integral (PI) controller, $K = k_p + \frac{k_I}{s}$, to increase the type with specifications (i) the gain crossover frequency has to be 100 rad/sec (ii) the phase margin has to be at least 40 degrees.

Problem 2



Feedback interconnection

Solution 1.

Solution: The breakpoint frequencies are shown on the bode-plot (next page).

They are at. 1rad/sec, 100 rad/sec and 10000rad/sec. All are poles

Thus, the transfer function is of the form

$$G(s) = \frac{A}{(s+1)(s+100)(s+10{,}000)}.$$

$$\Rightarrow G(0) = \frac{A}{10^2 \cdot 10^4} = \frac{A}{10^6}$$

but from the Bode plot $20\log_{10}|G(0)| = 20$

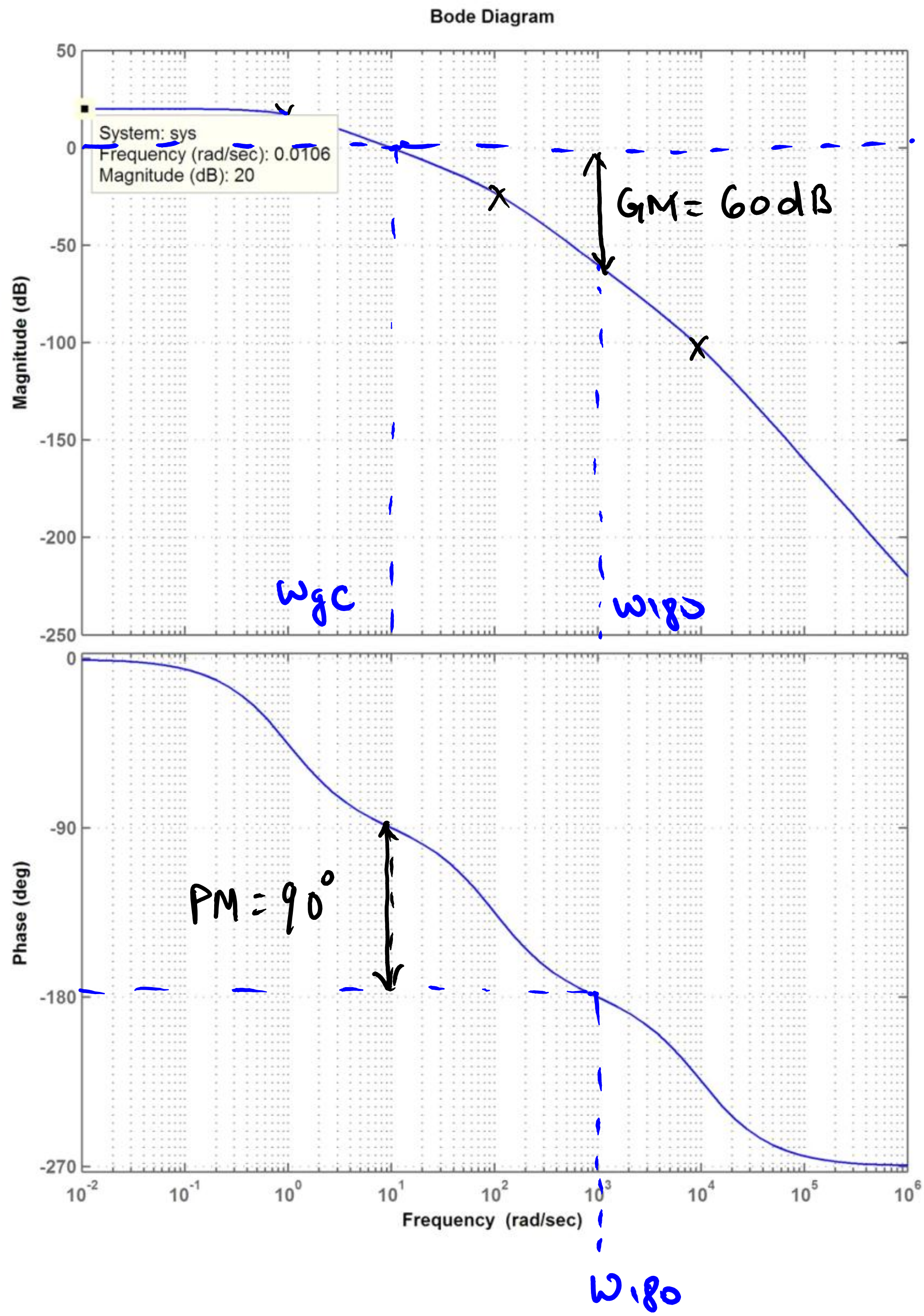
$$\Rightarrow |G(0)| = 10$$

Note that $\angle G(0) = 0$ $\therefore$ A is a positive constant and $\frac{A}{10^6} = 10 \Rightarrow A = 10^7$

$$\therefore G(s) = \frac{10^7}{(s+1)(s+100)(s+10000)} = \frac{10}{(s+1)\left(\frac{s}{100}+1\right)\left(\frac{s}{10^4}+1\right)}$$

is the transfer function

Consider the bode plot of a plant G as given below



Solution 2

Solution 2.

Solution: ω_{gc} = is the frequency at which

$$|G(j\omega_{gc})| = 1 \quad \text{or}$$

$$20\log_{10}|G(j\omega_{gc})| = 0 \text{ dB}$$

From the Bode plot

$\omega_{gc} =$ 10 rad/sec

Similarly the phase crossover ω_{180} is the frequency where

$$\angle G(j\omega_{180}) = -180$$

and this occurs at $\omega_{180} = 1000$ rad/sec

Determine the phase margin and the gain margin

Solution: The Gain margin is given by

$$GM = -20\log_{10}|G(j\omega_{180})|$$

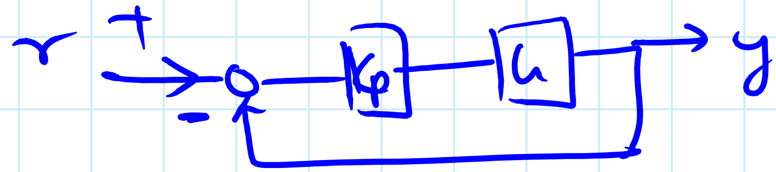
$$= 60\,dB$$

$$PM = 180 + \angle G(j\omega_{gc})$$

$$= 180 - 90 = 90 \text{ degrees}$$

[See Bode plot Earlier].

Solution 3.



3. With the controller $K=k_p$ a positive real constant, find the smallest value of k_p such that the interconnection shown is unstable. Use the gain margin to obtain the result.

Solution:

The value of K_p is given by

$$20\log_{10} k_p = GM$$

$$= 60\,dB.$$

$$\therefore \log_{10} k_p = \frac{60}{20} = 3$$

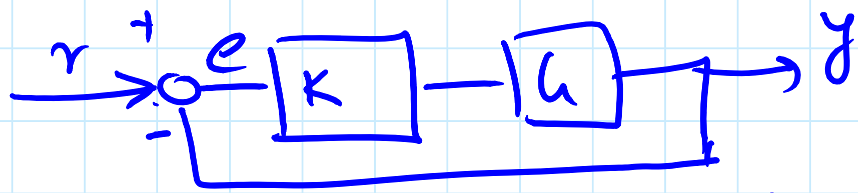
$$\Rightarrow k_p = 10^3 = 1000.$$

Thus, the smallest value of k_p that will destablized the feedback interconnection is 1000.

Solution to 4.

- With K=1, determine the steady state error due to a step input for the interconnection shown. Also, determine the steady state error due to a ramp input. Determine the type of the system.

Solution :



$L = KG = G.$

In this case transfer function from r to e is $\frac{1}{1+GK} = \frac{1}{1+L}$

and steady state error due to step is

$$\lim_{s\to 0} s\,e(s) = \lim_{s\to 0} s\left(\frac{1}{1+L}\right)\frac{1}{s}$$

$$= \lim_{s\to 0} \frac{1}{1+L}$$

$$= \frac{1}{1+k_p}\,; \quad k_p = L(0)$$

$L(0) = G(0)$ is such that

$20\log|G(0)| = 20\text{dB} \Rightarrow G(0) = 10$

and therefore $e_{ss} = \frac{1}{10+1} = 1/11 \approx 0.1$

when r is a ramp $e(s) = \left(\frac{1}{1+L}\right) 1/s^2$

and

e_{ss} due to ramp $= \lim_{s \to 0} s\, e(s)$

$$= \lim_{s \to 0} \frac{s}{(1+L)} \cdot \frac{1}{s^2}$$

$$= \lim_{s \to 0} \frac{1}{s + sL}$$

$$= \frac{1}{k_v};$$

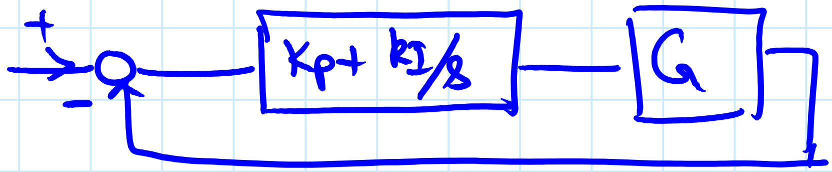
$k_v = \lim_{s \to 0} s\, L(s)$. As $k = 1$; $L = G$

and $k_v = \lim_{s \to 0} s\, G(s) = 0$

$\therefore$ e_{ss} due to ramp $= \infty$.

The system is Type 0; it does not track steps with zero steady state error.

Solution 5.



Design a Proportional Integral (PI) controller, K, to increase the type. Additional specification is that the gain crossover frequency has to 100 rad/sec and to have a PM of 40 degrees.

Solution: The PI controller is

$$K(s) = k_p + \frac{k_I}{s} = \frac{k_I}{s}\left[\frac{k_p}{k_I}s + 1\right]$$

$$= \frac{k_I}{s}\left[\frac{s}{k_I/k_p} + 1\right].$$

which has a break frequency at $\frac{k_I}{k_p}$.

→ $G(s)$ has a phase of $-135°$ at $\omega_{gcd} = 100$ r/sec

→ Thus, $PM_{have} = 180 - 135 = 45°$

safety margin

$PM_{desired} = 40 + 5$ also, thus, controller $K(s)$ cannot decrease the phase any further.

Thus, we choose $\frac{k_I}{k_p} \leq \frac{\omega_{gcd}}{10} = \frac{100}{10}$

$= 10$

Lets fix $\frac{k_I}{k_p} = 10$.

We also need to shift the gain crossover to 100 rad/sec. $= \omega_{gcd}$

$$\therefore \quad |L(j\omega_{gcd})| = 1$$

$$\Rightarrow \left|\left(k_p + \frac{k_I}{s}\right) G(s)\right| = 1 \quad \text{at } s = j\omega_{gcd}$$

$$\Rightarrow \left| k_p \left(1 + \frac{k_I}{k_p} \cdot \frac{1}{j\omega_{gcd}}\right)\right| |G(j\omega_{gcd})| = 1$$

$$\Rightarrow \left|k_p \left(1 - \left(\frac{k_I}{k_p} \frac{1}{100}\right) j\right| |G(j\omega_{gcd})| = 1$$

$$\Rightarrow k_p \left| 1 - j\frac{1}{10} \right| \, |G(j\omega_{gcd})| = 1$$

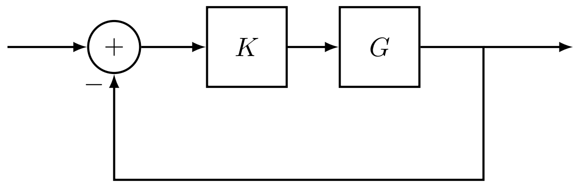
$$\Rightarrow k_p \sqrt{1 + \frac{1}{100}} \, |G(j \cdot 100)| = 1$$

$$\Rightarrow k_p = \frac{1}{|G(j \cdot 100)|} = \frac{1}{0.0741} = 13.5$$

$$\Rightarrow k_I = \frac{k_I}{k_p} \cdot k_p = 10 \cdot (13.5) = 135$$

$$\therefore \quad K(s) = 13.5 + \frac{135}{s}$$

Q3 [11pts]



Consider the unity gain loop depicted above, with open loop transfer function given by $KG(s) = K\frac{s+1}{s(s-1)}$. Let $K = k$ be a constant gain. Find the range of k that give phase margins of at least 30°.

Solution: The closed-loop poles are given by the roots of the polynomial $f(s) = s^2+(k-1)s+k$. It follows that the system is stable if and only if $k > 1$.

To find the phase margin, note that gain cross-over frequency is given by $\omega_{gc} = k$, since

$$|kG(j\omega_{gc})| = k\frac{|j\omega_{gc}+1|}{|j\omega_{gc}| \cdot |j\omega_{gc}-1|} = \frac{k}{\omega_{gc}} = 1.$$

Furthermore, the phase of $G(j\omega)$ is given by

$$\begin{aligned}\angle G(j\omega) &= \angle(j\omega+1) - 90° - \angle(j\omega-1)\\ &= \angle(j\omega+1) - 90° - (180° - \angle(j\omega+1))\\ &= 2\tan^{-1}(\omega) - 270°.\end{aligned}$$

It follows that the phase margin is given by

$$\varphi_{PM} = 2\tan^{-1}(k) - 90°.$$

Thus $\varphi_{PM} \geq 30°$ if and only if $\tan^{-1}(k) \geq (90° + 30°)/2 = 60°$, which holds if and only if $k \geq \sqrt{3}$.